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Entanglement generation in trapped atoms

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Abstract. The two atoms in the ion trap are entangled by the interaction with an external excited atom. The evolution of the entanglement is analytically derived without the decoherence. Considering the spontaneous decay from the environment, the evolution of the entanglement is similar to the damping Rabi oscillation. The generation of entanglement is induced by the dipole-dipole type interaction of atoms. It is found that the entanglement of two trapped atoms is robust with the uniform interaction with the external atom. The collective spontaneous emission from the coupling between the atoms may enhance the entanglement.

PACS. 03.65.Ud Entanglement and quantum nonlocality (e.g. EPR paradox, Bell's inequalities, GHZ states, etc.) – 03.67.Mn Entanglement production, characterization, and manipulation – 05.40.Ca Noise

QICS. 03.30.+e Entangling power of quantum evolutions – 02.40.+d Interaction with environment and decoherence – 15.10.En Ions: vibrational states

1 Introduction

The entanglement is of importance in the quantum information processing [1,2]. The resource of entanglement has been extensively used for quantum cryptography [3], quantum teleportation [4,5]. The experimental preparation of two entangled qubits has mainly focused on the nuclear magnetic resonance (NMR) and the cavity quantum electronic dynamics (Cavity-QED) [6]. It is necessary to qualify the entanglement. The relative entropy of entanglement [7,8] and the entanglement of formation [9] are basic measures for the bipartite systems. From the practical point of view, the decoherence of the entanglement is sometimes inevitable. By the interesting work [10], the external noise can give rise to the entanglement to some degree. With the consideration of heat reservoir, some protocols of entanglement generation have been put forward [11–15]. The development of the laser cooling and trapping technology provides us an efficient way to control the individual atom in traps [16,17]. Due to this point, it is of value to study the entanglement generation in trapped atoms.

In this paper, robust entanglement of two trapped atoms is generated by the interaction with an external excited atom. In Section 2, the evolution of the entanglement is obtained without the decoherence. In Section 3, the effects of atom spontaneous decay from the vacuum reservoir are studied. Nonuniform couplings among atoms are considered. In Section 4, the physical explanation of entanglement production is given by the analytical approximation. In Section 5, the exact analytical solution to the equation of motion is derived in the presence of the spontaneous emission. The effect of the collective spontaneous emission on the entanglement is considered. A discussion concludes the paper.

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2 Generation of two entangled trapped atoms

Our protocol of the entanglement production is realized in the trap. Two trapped two-level atoms a and b are initially prepared in the ground state $|g\rangle_a|g\rangle_b$. The external atom e is individually controlled and then interacts with atoms a and b. The coupling strengths of atoms a and b with e are denoted by g_a and g_b respectively. Without the impact of environment, the Hamiltonian of the total system is given by,

$$H = \sum_{i=a,b,e} \frac{\omega_i}{2}\sigma_i^z + \sum_{i=a,b} g_i(\sigma_i^+\sigma_e^- + h.c.), \quad (1)$$

where ω_i is the transition frequency of the ith atom, $\sigma_i^\pm$ is the transition operator satisfying $\sigma_i^+|g\rangle_i = |e\rangle_i$ and $\sigma_i^-|e\rangle_i = |g\rangle_i$. The symbol *h.c.* denotes the complex conjugate item. To simplify the expression, it is assumed that $\omega_a = \omega_b = \omega_e = \omega$, and $g_a = g_b = g$. In the interaction picture, equation (1) can be also expressed by

$$H_I = g\sum(\sigma_i^+\sigma_e^- + h.c.). \quad (2)$$

The system considered here resembles a system of three atoms coupled through the dipole-dipole nearest-neighbor interaction between the atoms. It is a simplified model of the Ising or XY Heisenberg chain system [18–20]. At the time t, the density matrix of the whole system is obtained by

$$\begin{aligned}\rho(t) &= U(t)\rho(0)U^{\dagger}(t)\\ &= U(t)|\psi(0)\rangle\langle\psi(0)|U^{\dagger}(t)\\ &= |\psi(t)\rangle\langle\psi(t)|,\end{aligned} \tag{3}$$

where $U(t) = \exp(-iH_I t)$ is the time evolution operator and $\rho(0)$ represents the initial state of the system. If the external atom e is initially prepared at the ground state $|g\rangle_e$, the total system will keep unchanged at the state $|g\rangle_a|g\rangle_b|g\rangle_e$. In this case, the state of two trapped atoms a and b is unentangled. When atom e is initially excited, the state of the whole state is $|\psi(0)\rangle = |g\rangle_a|g\rangle_b|e\rangle_e$. By the expansion of equation (3), $\rho(t)$ is written as,

$$\begin{aligned}|\psi(t)\rangle =& \cos\sqrt{2}gt|g\rangle_a|g\rangle_b|e\rangle_e\\ &-\frac{i}{\sqrt{2}}\sin\sqrt{2}gt(|e\rangle_a|g\rangle_b|g\rangle_e + |g\rangle_a|e\rangle_b|g\rangle_e).\end{aligned} \tag{4}$$

In the space of $\{|ee\rangle_{ab}, |eg\rangle_{ab}, |ge\rangle_{ab}, |gg\rangle_{ab}\}$, the reduced density matrix of atoms a and b is,

$$\rho_{ab}(t) = Tr_e[\rho(t)] = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & v & v & 0\\ 0 & v & v & 0\\ 0 & 0 & 0 & y \end{pmatrix}, \tag{5}$$

where the elements are $v = \frac{1}{2}\sin^2\sqrt{2}gt$, $y = \cos^2\sqrt{2}gt$. The concurrence measure is used to evaluate the entanglement [9],

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{6}$$

where $\{\lambda_i, |i = 1, 2, 3, 4\}$ are the square root of eigenvalues of the non-Hermitian matrix $\rho\tilde{\rho}$ with $\tilde{\rho} = (\sigma^y \otimes \sigma^y)\rho^*(\sigma^y \otimes \sigma^y)$ in the decreasing order. According to equation (6), the entanglement of atoms a and b is,

$$C = 2|v| = \sin^2\sqrt{2}gt. \tag{7}$$

It is found that entanglement of atoms a and b follows the sine periodic evolution. At the time $gt = \frac{2n+1}{2\sqrt{2}}\pi, (n = 0, 1, 2, ...)$, the concurrence arrives at the maximum 1. If $|g\rangle_e$ is detected, the state ρ_{ab} is just at the maximally entangled state with the form of $|\psi^+\rangle_{ab} = \frac{1}{\sqrt{2}}(|e\rangle_a|g\rangle_b + |g\rangle_a|e\rangle_b)$.

3 Spontaneous decay and nonuniform couplings

The former case of entanglement generation is discussed without the effects of the environment. As a matter of fact, the entanglement decoherence exists in some conditions [21]. It is assumed that $2\Gamma_i$ $(i = a, b, e)$ is the spontaneous decay rate. Considering the vacuum reservoir, the master equation governing the time evolution of the global system is given by,

$$\dot{\rho} = -i[H_I, \rho] + L(\rho), \tag{8}$$

where H_I is the Hamiltonian in the interaction picture and the Liouvillian $L(\rho)$ denotes the spontaneous decay due to the vacuum reservoir. By controlling the external atom e, the coupling strengths g_a and g_b may be nonuniform, i.e. $g_a \neq g_b$. The Hamiltonian H_I can be expressed as,

$$H_I = g(1+\delta)(\sigma_a^+\sigma_e^- + h.c.) + g(1-\delta)(\sigma_b^+\sigma_e^- + h.c.), \tag{9}$$

where $g = \frac{g_a+g_b}{2}$ is the mean value of the couplings and $\delta = \frac{g_a+g_b}{g_a-g_b}$ is nonuniform coefficient of the couplings. The Liouvillian item $L(\rho)$ is

$$L(\rho) = \sum_{i=a,b,e} \Gamma_i(2\sigma_i^-\rho\sigma_i^+ - \sigma_i^+\sigma_i^-\rho - \rho\sigma_i^+\sigma_i^-). \tag{10}$$

The analytical solution of equation (8) is extremely tedious. The entanglement for general parameters is numerically illustrated, and the approximate analytical result is obtained.

The evolution of the entanglement is plotted in Figures 1a and 1b. In Figure 1a, the concurrence of atoms a and b is plotted as a function of the spontaneous decay Γ and time t when the coupling is uniformly distributed. The damping Rabi oscillation of the concurrence appears when $\Gamma \neq 0$ at any time. The concurrence is monotonically decreased with the increase of Γ. It is demonstrated that the spontaneous decay destroys the generation of entanglement. In Figure 1b, the concurrence is plotted as a function of the nonuniform coefficient δ and time t. It is seen that the concurrence is always maximal for $\delta = 0$. The peak values of C occur and are decreased with the time. Meanwhile, the concurrence is zero at some time and are gradually increased across those time. The evolution behavior of the concurrence is similar to the Rabi oscillation. It is found that the entanglement generation is robust when the couplings are uniform and the entanglement is zero at some time. The effects of the spontaneous decay Γ and the non-uniform coefficient δ on the concurrence C are shown in Figure 1c. It is seen that the concurrence C is decreased monotonically with the increase of Γ, and the concurrence C is maximal when $\delta = 0$. From Figure 1, it is found that the behavior of the concurrence is like the Rabi oscillation. The amplitude of oscillation is modulated by the dissipative item, $L(\rho)$. Robust generation of entanglement occurs when the interactions between atoms are uniform.

4 Approximate analysis

To understand the origin of entanglement generation in two trapped atoms, the analytical approximation is investigated. According to equation (5), the concurrence for

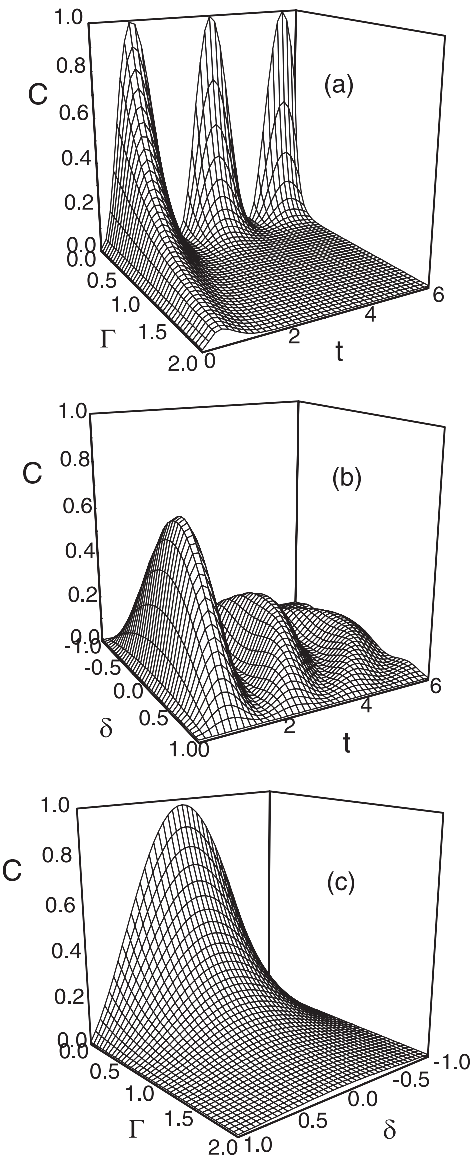

Fig. 1. The concurrence C as a function of Γ, δ and t for $\omega_a = \omega_b = \omega_e = \omega, g = 1.0$, and $\Gamma_a = \Gamma_b = \Gamma_e = \Gamma$ when (a) the concurrence C as a function of the spontaneous decay rate Γ and the time t when $\delta = 0.0$; (b) the concurrence C as a function of the non-uniform coefficient δ and the time t when $\Gamma = 0.2$; (c) the concurrence C as a function of Γ and δ when $t = 1.0$.

$\Gamma = 0$ is decided by the element $\langle ge|\rho|eg\rangle$. From the physical point of view, the element denotes the transition possibility between two atoms from the upper level $|e\rangle_a$ (or$|e\rangle_b$) to the low level $|g\rangle_b$ (or $|g\rangle_a$). With regard to the spontaneous decay from vacuum reservoir, the approximation result is obtained by the perturbation method. The density matrix $\rho(t)$ expanded at the time $t = 0$ is given,

$$\rho(t) = \rho(0) + t\frac{\partial\rho}{\partial t}|_{t=0} + \frac{t^2}{2!}\frac{\partial^2\rho}{\partial t^2}|_{t=0} + \frac{t^3}{3!}\frac{\partial^3\rho}{\partial t^3}|_{t=0} + ... \quad (11)$$

where $\frac{\partial^i\rho}{\partial t^i}$ is the ith-order derivative of the density matrix. At the limit of $t \ll 1$, the above expansion may be truncated at the third-order approximation. The result is given by,

$$\begin{aligned}\frac{\partial\rho}{\partial t} &= -i[H, \rho(0)] + L(\rho(0)),\\ \frac{\partial^2\rho}{\partial t^2} &= -i[H, \frac{\partial\rho}{\partial t}] + L\left(\frac{\partial\rho}{\partial t}\right),\\ \frac{\partial^3\rho}{\partial t^3} &= -i[H, \frac{\partial^2\rho}{\partial t^2}] + L\left(\frac{\partial^2\rho}{\partial t^2}\right). \end{aligned} \quad (12)$$

If the initial state is $|g\rangle_a|g\rangle_b|e\rangle_e$, the reduced density matrix of atoms a and b is

$$\rho_{ab}(t) = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & w & v & 0\\ 0 & v & z & 0\\ 0 & 0 & 0 & x \end{pmatrix}, \quad (13)$$

where $w = g^2(1+\delta^2)t^2 - \frac{2}{3}g^2\Gamma(3\delta^2+4\delta+3)t^3$, $v = g^2(1-\delta^2)t^2 - 2g^2\Gamma(1-\delta^2)t^3$, $z = g^2(1-\delta^2)t^2 - \frac{2}{3}g^2\Gamma(\delta^2-4\delta+3)$ and $x = 1 - 2\Gamma^2t^2 + 4g^2\Gamma(1+\frac{2}{3}\delta^2)t^3$. According to equation (6), the concurrence C can be expressed as

$$C = 2|v| = 2g^2(1-\delta^2)t^2 - 4g^2\Gamma(1-\delta^2)t^3. \quad (14)$$

In equation (14), the concurrence is optimal when the couplings are uniform with $\delta = 0$. The values of the concurrence are decreased with the increase of Γ. Since the transition probability $\langle\sigma_a^+\sigma_b^-\rangle$ can determine the behavior of the concurrence, the time evolution of $\langle\sigma_a^+\sigma_b^-\rangle$ is plotted in Figure 2. The value of $\langle\sigma_a^+\sigma_b^-\rangle$ is plotted in Figure 2a when there is no spontaneous decay with $\Gamma = 0$. From Figure 2a, it is seen that the transition possibility $\langle\sigma_a^+\sigma_b^-\rangle$ shows periodic behavior of sine function. If the spontaneous decay exists in the system with $\Gamma \neq 0$ as shown in Figure 2b, the behavior of $\langle\sigma_a^+\sigma_b^-\rangle$ is similar to the damping Rabi oscillation. It follows that the concurrence of atoms a and b is mainly determined by the transition possibility. If the existence of the transition possibility is assured, the entanglement generation in trapped atoms can be realized.

5 Exact solution

If only one atom is excited, the equation of motion for atomic operators can be exactly solved [22]. All the elements for the density matrix of three atoms can also be obtained. The exact evolution of the entanglement can be completely demonstrated by the analytical solutions to the concurrence. The equation of motion [23,24] of the atomic operators Q is expressed by

$$\begin{aligned}\langle\dot{Q}\rangle = &-i\sum_{i=a,b} g_i\langle[\sigma_i^+\sigma_e^- + \sigma_i^-\sigma_e^+, Q]\rangle\\ &+ \sum_{i=a,b,e}\Gamma_i\langle 2\sigma_i^+Q\sigma_i^- - \sigma_i^+\sigma_i^-Q - Q\sigma_i^+\sigma_i^-\rangle, \end{aligned} \quad (15)$$

where $\langle Q\rangle$ is the expection value of the atomic operator Q for the density matrix of three atoms ρ. Here the simple

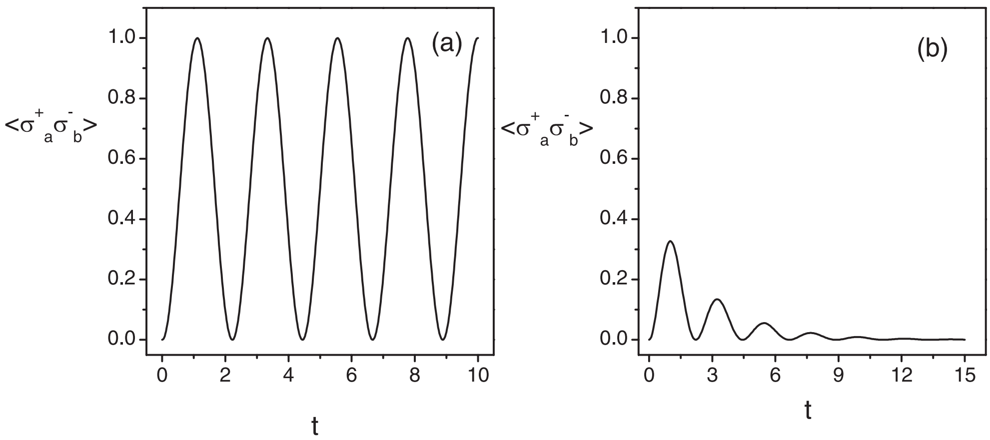

Fig. 2. The transition possibility $\langle \sigma_a^+ \sigma_b^- \rangle$ as a function of the time t is plotted for $\omega_a = \omega_b = \omega_e = \omega$, and $\Gamma_a = \Gamma_b = \Gamma_e = \Gamma$ when: (a) $\delta = 0$ and $\Gamma = 0$; (b) $\delta = 0$ and $\Gamma = 0.2$.

case of $g_a = g_b = g$ and $\Gamma_a = \Gamma_b = \Gamma_e = \Gamma$ is considered. In the initial condition of only one atom excited, i.e. $\rho(0) = |gge\rangle\langle gge|$, the reduced density matrix of the atoms a and b can be written as

$$\rho_{ab}(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & R & R & 0 \\ 0 & R & R & 0 \\ 0 & 0 & 0 & S+U \end{pmatrix}. \quad (16)$$

The above elements can be calculated by the equation of motion for some atomic operators

$$\begin{aligned} R &= \langle \sigma_a^+ \sigma_b^- \sigma_e^- \sigma_e^+ \rangle, \\ S &= \langle \sigma_a^- \sigma_a^+ \sigma_b^- \sigma_b^+ \sigma_e^+ \sigma_e^- \rangle, \\ U &= \langle \sigma_a^- \sigma_a^+ \sigma_b^- \sigma_b^+ \sigma_e^- \sigma_e^+ \rangle. \end{aligned} \quad (17)$$

These elements obey equation (15) and can be given by the following differential equations

$$\begin{aligned} \dot{R} &= -2\Gamma R - 2gX, \\ \dot{X} &= -2\Gamma X + g(2R - S), \\ \dot{S} &= -2\Gamma S + 4gX, \\ \dot{U} &= 2\Gamma(2R + S), \end{aligned} \quad (18)$$

where $X = i\langle \sigma_a^- \sigma_a^+ \sigma_b^- \sigma_e^+ \rangle$ also satisfies equation (15). The initial conditions of equation (18) are $R(0) = X(0) = U(0) = 0$ and $S(0) = 1$. Then, the exact analytical solution to the elements for the density matrix can be obtained by

$$\begin{aligned} R &= \exp(-2\Gamma t)\sin^2 gt, \\ X &= -\frac{1}{2}\exp(-2\Gamma t)\sin 2gt, \\ S &= \exp(-2\Gamma t)\cos 2gt, \\ U &= 1 - \exp(-2\Gamma t). \end{aligned} \quad (19)$$

Therefore, using equation (6), the concurrence of C can be exactly expressed by $C(t) = 2|R| = 2\exp(-2\Gamma t)\sin^2 gt$.

For the strong dipole-dipole interactions, the collective spontaneous emissions between the atoms need to be included in the master equation [25]. The master equation

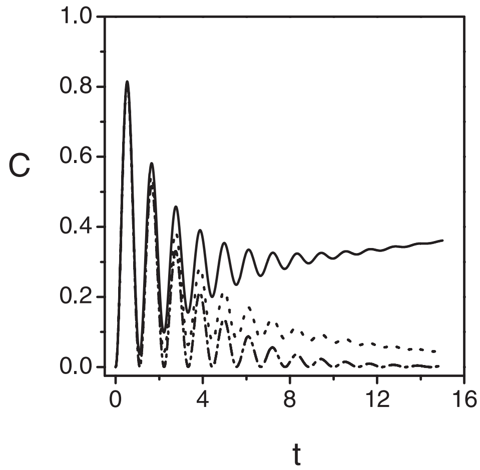

Fig. 3. The concurrence C is plotted as a function of the time t when $g_a = g_b = g = 2.0$, $\Gamma_a = \Gamma_b = \Gamma_e = \Gamma = 0.2$. The collective spontaneous emission Γ_C is chosen to be 0.0, 0.1, 0.15 (from bottom to top).

with the collective effects can be expressed by

$$\begin{aligned} \dot{\rho} &= -i[H_I, \rho] + L_c(\rho), \\ L_c(\rho) &= \sum_{i=a,b,e} \Gamma(2\sigma_i^- \rho \sigma_i^+ - \sigma_i^+ \sigma_i^- \rho - \rho \sigma_i^+ \sigma_i^-) \\ &\quad + \sum_{j=a,b} \Gamma_c(2\sigma_e^- \rho \sigma_j^+ - \sigma_j^+ \sigma_e^- \rho - \rho \sigma_j^+ \sigma_e^-) \\ &\quad + \sum_{k=a,b} \Gamma_c(2\sigma_k^- \rho \sigma_e^+ - \sigma_e^+ \sigma_k^- \rho - \rho \sigma_e^+ \sigma_k^-). \end{aligned} \quad (20)$$

Where the collective spontaneous emission rate is $2\Gamma_c$. The concurrence C is plotted in Figure 3 when the collective spontaneous emission rate Γ_c is varied. The effect of Γ_c on the entanglement is clearly shown. From Figure 3, it is seen that the entanglement will oscillate while the envelop of C will decrease to zero with time t if the collective effect is neglected. When the collective spontaneous emission is included in the system, the envelop of C is decreased and then increased with time t. The generation of the entanglement between atoms a and b is enhanced to some degree for large values of Γ_c and t. Especially after a certain time, the improvement from the collective effects is more apparent. It is found that the collective effect can enhance the generation of the entanglement.

6 Discussion

The entanglement of two trapped atoms is generated by the interaction with an external excited atom. If the external atom is detected at $|g\rangle_e$, the trapped atoms are maximally entangled at the Bell state $|\psi^+\rangle_{ab}$. Considering the spontaneous decay from the vacuum reservoir, the behavior of entanglement is similar to the damping Rabi oscillation. The amplitudes of the concurrence are modulated by the dissipative item $L(\rho)$. The evolution behavior of the entanglement is mainly dependent on the transition possibility between two trapped atoms. It is found that the generation of entanglement is optimal when the couplings are uniform. The collective spontaneous emission from the couplings can enhance the generation of the entanglement at relatively long time.

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